Stationary-to-stationary solutions in relativity theory

1975 J. Phys. A: Math. Gen. 8 506

(http://iopscience.iop.org/0305-4470/8/4/011)

J. Phys. A: Math. Gen., Vol. 8, No. 4, 1975. Printed in Great Britain. © 1975

Stationary-to-stationary solutions in relativity theory

A Sackfield
Trent Polytechnic, Nottingham, NG1 4BU, UK

Received 14 November 1974

Abstract. Stationary axially symmetric solutions claimed to be original by Kloster, Som and Das are in fact known. A technique for derivation of the class of solutions is given.

1. Introduction

In a recent paper Kloster *et al* (1974) display a number of very interesting results and theorems concerning the stationary gravitational field. However, in their specialization to axially symmetric stationary fields they claim discovery of a new class of solutions requiring fr^{-1} functionally dependent on w (terms defined below). This dependence however is precisely that required of the Ehlers (1965) solutions. We give a simple mapping for obtaining the Ehlers solutions from other known solutions.

2. Field equations

In its usual form the stationary axially symmetric metric may be written

$$ds^2 = -f^{-1}[e^{2k}(dr^2+dz^2)+r^2\,d\phi^2]+f(dt-w\,d\phi)^2, \tag{1}$$

f, k and w being functions of r and z only. The resultant vacuum field equations for f and w are

$$\begin{aligned} &f\nabla^2 f = (\nabla f)^2 - f^4 r^{-2}(\nabla w)^2, \\ &f(w_{11}+w_{22}-r^{-1}w_1)+2\nabla f.\nabla w = 0 \end{aligned} \tag{2}$$

where subscripts 1 and 2 denote differentiation with respect to r and z respectively and ∇, ∇^2 are the usual Euclidean operators in the flat background cylindrical coordinate system (r, ϕ, z). Once f and w are determined the final field equations may be solved for k in terms of quadratures.

Now the equations (2) have been recast (Ernst 1968) into the form

$$f\nabla^2 f = (\nabla f)^2 - (\nabla\xi)^2, \qquad f\nabla^2\xi = 2\nabla f.\nabla\xi \tag{3}$$

where

$$f^2\nabla w = r\hat{n}_\wedge\nabla\xi \tag{4}$$

and $\hat{n}$ is a unit vector in the ϕ direction. It is then a remarkable fact that

$$f\to rf^{-1}, \qquad w\to i\xi \tag{5}$$

maps (2) into (3). Thus a solution (f, w) of (2) may be mapped into a solution (f, ξ) of (3) and hence by (4) into a new solution of (2). Now it is well known (eg Sackfield 1971) that the imposition of a functional relationship between f and ξ implies the Papapetrou (1953) solutions. With this requirement the mapping (5) thus generates a new class with w a function of fr^{-1}: *the Ehlers class.*

References

Ehlers J 1965 *Proc. Int. Conf. on Relativistic Theories of Gravitation, London* vol 2 p 15
Ernst F J 1968 *Phys. Rev.* **167** 1175
Kloster S, Som M M and Das A 1974 *J. Math. Phys.* **15** 1096
Papapetrou A 1953 *Ann. Phys., Lpz* **12** 309
Sackfield A 1971 *Proc. Camb. Phil. Soc.* **70** 89